**Econometrics 3 (ECOM90013) Assignment 3**

**Let denote a simple random sample from a population with probability density function**

1. **(7 marks) Show that the sample mean is a consistent estimator of .**

**Hint: first derive the mean of the population and then remember that laws of large numbers are your friends.**

First, we derive the mean of the population:

Using the power rule of integration, noting by assumption:

Hence, we get:

Therefore:

The Weak Law of Large Numbers (WLLN) can only be applied in the instances of finite variance. Hence, we confirm this:

Using the power rule again, applied to the square term, we first evaluate for first term:

Applying the same logic as previously, this simplifies to:

Now we can evaluate this with the second term for the variance:

Although un-simplified, this expression tells us there is a finite variance. Therefore, we can apply the WLLN to it.

Therefore, as this term is bounded according to the WLLN, we can say the sample average is a consistent estimator for the mean. Or, more formally:

1. **(1 mark) Derive a consistent method of moments estimator, say, for .**

To derive the consistent method of moments estimator, , for we use the observed sample mean to solve for our estimator rather than computing the population mean. Therefore, we need to take our definition of the sample average and solve for our unknown.

Now we simply need to solve for :

Therefore:

1. **(1 mark) Specify the log-likelihood function for this sample.**

The log likelihood function is:

1. **(3 marks) Derive the maximum likelihood estimator, say, for and prove that it is, indeed a *maximum* likelihood estimator.**

To make notation simpler:

Therefore:

First, we differentiate the log-likelihood function with respect to :

Now we formally state the first order condition:

Hence our maximum likelihood estimator is given as:

Now to verify this is actually a maximum by verifying the Hessian is negative:

As it’s assumed this Hessian must be negative, telling us the log-likelihood is concave and the maximum likelihood estimator is actually a maximum.

1. **(2 marks) Derive the Fisher information for the sample.**

The Fisher information is the negative expected value of the Hessian:

1. **(2 marks) Suppose that someone wishes to test the null hypothesis against the alternative that . State the true population density function and describe in words the implication for the population when this null hypothesis when this null hypothesis is true.**
2. **(12 marks) Derive the likelihood ratio, Lagrange multiplier and Wald tests for the hypotheses of Question 6. In each case provide the decision rule that you would use in practice to apply the test, including any critical value(s) you may need.**
3. **(6 marks) Without appeal to the generic properties of maximum likelihood estimators, prove that is consistent for .**